Squeeze Thm for Functional Limits.

If f, g, h are real-value fcns on a set A and c is a limit pt of A; and if

$$f(x) \leq g(x) \leq h(x) \quad \text{for all } x \in A,$$

and if $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = L$,

Then $\lim_{x \to c} g(x)$ exists and $= L$.

Continuous Functions.

Defn Let $f: A \to \mathbb{R}$ be a continuous function, and let $c \in A$.

We say that f is continuous at c if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $|x - c| < \delta$ and $x \in A$, then $|f(x) - f(c)| < \varepsilon$.

Equivalent Defns:

① Limit defn.

f is continuous at c

$\iff$ Either c is an isolated pt of A

or c is a limit pt of A

and $\lim_{x \to c} f(x) = f(c)$.

---

② Sequential defn.

f is continuous at c.

$\iff$ For every sequence (x_n) in A s.t.

$\lim x_n = c$,

we have $\lim f(x_n) = f(c)$.

__Defn__ A function $f: A \to \mathbb{R}$ is __continuous__ if $\forall c \in A$, f is continuous at c.

__Topological Defn of Continuous.__

$f: A \to \mathbb{R}$ is __continuous__ if the inverse image of every open set is open.

Note if $B \subseteq \mathbb{R}$ is a set, the inverse image of B is

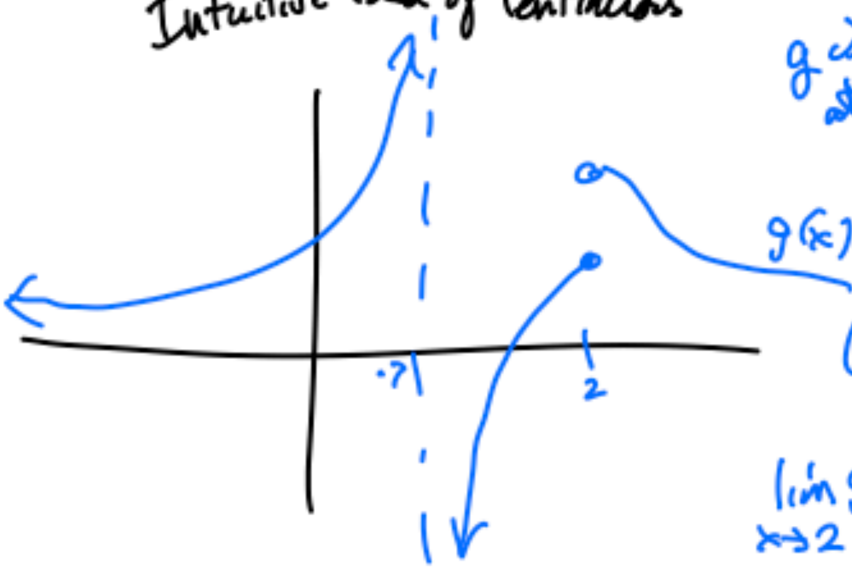
$$f^{-1}(B) = \{x \in A : f(x) \in B\}.$$

eg $f(x) = x^2$

$B = [2,5)$

$$\begin{aligned} f^{-1}(B) &= \{x : x^2 \in B\} \\ &= \{x : 2 \le x^2 < 5\} \\ &= \{x : \sqrt{2} \le x < \sqrt{5} \text{ or } -\sqrt{2} \ge x > -\sqrt{5}\} \\ &= [\sqrt{2}, \sqrt{5}) \cup (-\sqrt{5}, -\sqrt{2}]. \end{aligned}$$

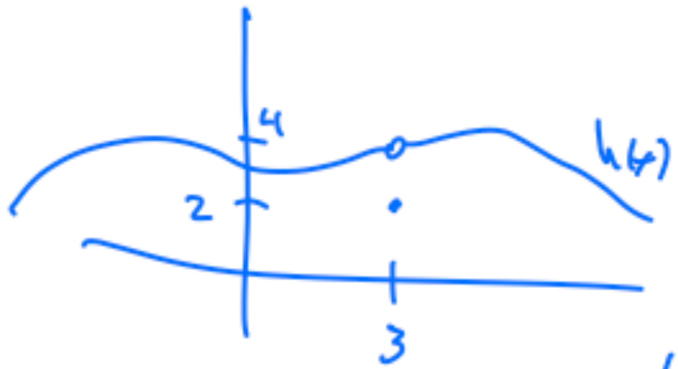
Intuitive idea of Continuous



g is continuous at every point of its domain except $x = 2$.

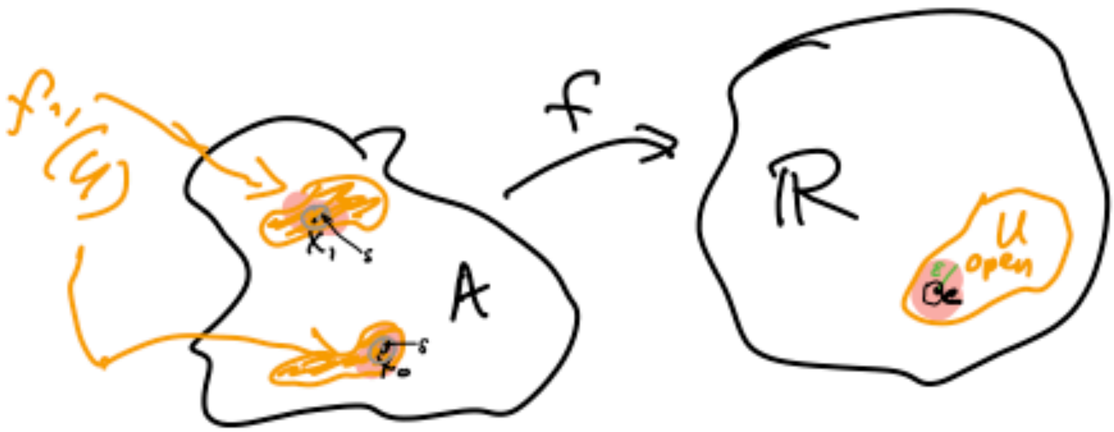
(Domain $= \mathbb{R} \setminus \{-7\}$)

$\lim_{x \to 2} g(x)$ does not exist.



$\lim_{x\to 3} h(x) = 4$
but $h(3)=2$.
So h is not continuous at 3.

Intuitive idea of the topological defn.



$f(x_0)=c$
$f(x_1)=c$

Example – Show that $f(x)=x^2-x+1$ is continuous on $\mathbb{R}$.

Scratchwork. We need $\forall c\in\mathbb{R}$ $|x-c|<\delta \implies |f(x)-f(c)|<\varepsilon$

want $\left|(x^2-x+1)-(c^2-c+1)\right|<\varepsilon$

$$|x^2-c^2-x+c|<\varepsilon$$

$$|x^2-c^2-x+c| \underset{\text{triangle}}{\leq} |x^2-c^2|+\underset{\text{force}}{|x-c|}<\varepsilon$$

$$|x^2-c^2+-(x-c)| \leq$$

$$|(x-c)(x+c)|+|x-c|<\varepsilon$$

$$|x-c|\,|x+c|+|x-c|<\varepsilon$$

$$|x-c|\left(|x+c|+1\right)<\varepsilon$$

δ can't depend on x.

except $|x-c|<\delta$ $\quad |x-c|<\dfrac{\varepsilon}{|x+c|+1}$

$$\delta=\min\{1,\ \_\_\}\qquad \frac{\varepsilon}{\_\_+1}<\frac{\varepsilon}{|x+c|+1}$$

$$|x-c|<1$$

$$c-1<x<c+1$$

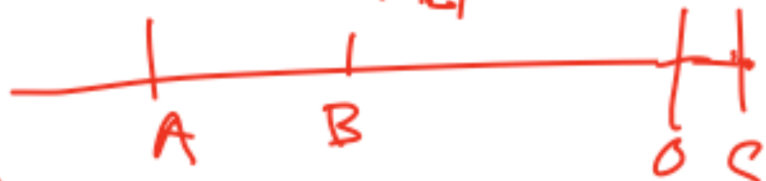
$$2c-1<x+c<2c+1$$

$x-1<c<x+1$

$A < B < C$ $\qquad -7 < \underset{-6}{-2} < 3$

wait $|B| \le \underline{\max}\{|A|, |C|\}$

$\le |A| + |C|$



$\delta \searrow$

$$|x-c| < \frac{\varepsilon}{|2c-1| + |2c+1| + 1} \le \frac{\varepsilon}{|x+c|+1}$$

Show that $f(x) = x^2 - x + 1$ is continuous on $\mathbb{R}$.

<u>Actual proof</u>: $\forall c \in \mathbb{R}$,

$\forall \varepsilon > 0$, let $\delta = \min\left\{1, \frac{\varepsilon}{|2c-1| + |2c+1| + 1}\right\}$.

Then, if $|x-c| < \delta$,

we have $|x-c| < 1$

$\Rightarrow c-1 < x < c+1$

$\Rightarrow 2c-1 < x+c < 2c+1$

$\Rightarrow |x+c| < |2c+1| + |2c-1|$

Thus $\delta = \frac{\varepsilon}{|2c-1| + |2c+1| + 1} < \frac{\varepsilon}{|x+c|+1}$

Hence, if $|x-c| < \delta$,

$$|x-c| < \frac{\varepsilon}{|x+c|+1}$$

$$\Rightarrow |x+c||x-c| + |x-c| < \varepsilon$$

$$\Rightarrow |x^2-c^2| + |x-c| < \varepsilon$$

By triangle $\leq$, $\Rightarrow |x^2-c^2-x+c| < \varepsilon$

$$\Rightarrow |(x^2-x+1)-(c^2-c+1)| < \varepsilon.$$

$\therefore$ f is continuous at c.

$\therefore$ f is continuous on $\mathbb{R}$. ▨

Algebraic Continuity Thm.

Suppose f, g are continuous at $c \in A$ with $f, g : A \to \mathbb{R}$, and let $r, s \in \mathbb{R}$.

Then ① $f(x) \pm g(x)$ is continuous at c.

② $f(x)\cdot g(x)$ is continuous at c.

③ $\dfrac{f(x)}{g(x)}$ is continuous at c if $g(c)\neq 0$.

④ $\sqrt{f(x)}$ is cont. at c if $f(x)\geq 0\ \forall x\in A$.

⑤ $rf(x)+sg(x)$ is continuous at c.

<u>Example Proof</u>: ③.

Want $|x-c|<\delta \Rightarrow \left|\dfrac{f(x)}{g(x)}-\dfrac{f(c)}{g(c)}\right|<\varepsilon$ want

Know

$|f(x)-f(c)|<\varepsilon_1$

$|g(x)-g(c)|<\varepsilon_2$

$g(c)\neq 0$.

$$\left|\frac{f(x)}{g(x)}-\frac{f(c)}{g(c)}\right|<\varepsilon$$

$$\left|\frac{f(x)g(c)-g(x)f(c)}{g(x)g(c)}\right|<\varepsilon.$$

let $\varepsilon_2\leq \dfrac{|g(c)|}{2}$

so that $g(x)$

$\dfrac{|g(c)|}{2}\leq |g(x)|\leq \dfrac{3}{2}|g(c)|$

$$\frac{|f(x)g(c)-g(x)f(c)|}{\frac{|g(c)|}{2}\,|g(c)|}<\varepsilon$$

$$|f(x)\,g(c) - g(x)\,f(c)| < \frac{\varepsilon\,|g(c)|^2}{2}.$$

$$|f(x)g(c) - f(c)g(c) + f(c)g(c) - g(x)f(c)| < \frac{\varepsilon|g(c)|^2}{2}$$

$$\leq$$

$$|f(x)-f(c)|\,|g(c)| + |f(c)|\,|g(c)-g(x)| < \frac{\varepsilon|g(c)|^2}{2}$$

$$< \frac{\varepsilon}{2|g(c)|}\frac{|g(c)|^2}{2} \qquad < \frac{\varepsilon}{2|f(c)|}\frac{|g(c)|^2}{2}$$

Choose δ s.t. $|g(x)-g(c)| < \frac{|g(c)|}{2}$

$|f(x)-f(c)| < \frac{\varepsilon}{4}|g(c)|$

$|g(x)-g(c)| < \frac{\varepsilon}{4}\frac{|g(c)|^2}{|f(c)|}$ if $f(c) \neq 0$.

Proof that $\frac{f(x)}{g(x)}$ is continuous at c if $g(c) \neq 0$ & f & g are continuous at c:

$\forall \varepsilon > 0$, chose $\delta > 0$ s.t.

$$|g(x) - g(c)| < \min\left\{\frac{|g(c)|}{2}, \frac{\varepsilon|g(c)|^2}{4|f(c)|} \text{ if } f(c) \neq 0\right\}$$

and $|f(x)-f(c)| < \frac{\varepsilon}{4}|g(c)|$.

Then $\frac{|g(c)|}{2} \leq |g(x)| \leq \frac{3|g(c)|}{2}$

and

$$|f(x)-f(c)||g(c)| + |f(c)||g(c)-g(x)|$$

$$< \frac{\varepsilon}{4}|g(c)|^2 + \begin{cases} 0 & \text{if } f(c)=0 \\ \frac{\varepsilon}{4}|g(c)|^2 \end{cases}$$

$$\leq \frac{\varepsilon}{2}|g(c)|^2$$

Then by Triangle $\leq$,

$$|f(x)g(c) - f(c)g(c) + f(c)g(c) - f(c)g(x)| < \frac{\varepsilon}{2}|g(c)|^2$$

$$\Rightarrow |f(x)g(c) - f(c)g(x)| < \frac{\varepsilon}{2}|g(c)|^2$$

$$\Rightarrow \frac{|f(x)g(c) - f(c)g(x)|}{\frac{1}{2}|g(c)|^2} < \varepsilon$$

since $\frac{|g(c)|}{2} \leq |g(x)|$

$$\Rightarrow \left|\frac{f(x)g(c) - f(c)g(x)}{g(x)g(c)}\right| < \varepsilon$$

$$\Rightarrow \left|\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}\right| < \varepsilon. \quad \blacksquare$$

Cor Rational fcns & other fcns involving $\sqrt{\ }$ are continuous on their domains.

Another useful fact:

Thm The composition of continuous fcns is continuous.

[That is, If $f : A \to B$ is continuous at $a \in A$ and $g : B \to C$ is continuous at $f(a) \in B$, then $(g \circ f) : A \to C$ is continuous at a.].

Pf. With given, $\forall \varepsilon > 0$, choose δ_1 s.t. $|y - f(a)| < \delta_1, y \in B \Rightarrow |g(y) - g(f(a))| < \varepsilon$.
Choose $\delta > 0$ s.t. $|x - a| < \delta, x \in A \Rightarrow |f(x) - f(a)| < \delta_1$

$$\Rightarrow |g(f(x)) - g(f(a))| < \varepsilon. \quad \blacksquare$$